J. Fluid Mech. (1974), *vol.* 66, *part* 4, *pp.* 817–829

Printed in Great Britain

High-intensity light propagation and induced natural laminar flow

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(Received 15 February 1974)

The objective of this paper is to describe approximately the coupled steady-state processes of light propagation and induced laminar incompressible fluid flow in the case of natural convection.

For the case of a homogeneous fluid and under the assumptions that light energy is instantaneously transformed into heat and that the induced velocities are not too large, it is reasonable to use the boundary-layer equations to describe the induced natural flow. These equations are augmented by the conservation of energy equation. The velocity, temperature and intensity functions are expected to exhibit similarity properties.

A high-intensity light beam with a given rotationally symmetric Gaussian initial intensity distribution is propagating vertically upwards into a fluid initially at rest. The fluid characteristics are assumed to be constant. A stream function is introduced to satisfy the conservation of mass equation. The conservation of momentum equation leads to conditions on the unknown functions involved in the stream function. Additional conditions follow from the conservation of energy equation, which involves the local light intensity as a driving term.

Under the assumptions made, self-defocusing (thermal blooming) will occur. The main results are an exponential increase of the boundary-layer thickness and an exponential decrease of temperature and of light intensity due to the blooming effect in addition to the exponential decrease due to absorption.

1. Introduction

Light propagating through an absorbing fluid creates local changes in temperature and, consequently, in density. Therefore, a fluid flow will be established by the buoyancy force. The resulting redistribution of temperature will affect the local light intensity distribution. Thus, a complicated coupled process of propagation of light, distribution of temperature and the corresponding flow in the fluid will take place.

The aspects of this process become the more complicated the more one looks at the details of the absorption process, i.e. the interaction of intense electromagnetic fields with the molecules of the fluid (which is responsible for their thermal agitation), and at the behaviour of the index of refraction. If the relaxation process is rapid (as for water, for example, under 10·6 μm radiation), the absorbed energy is immediately transformed into heat, i.e. an instantaneous increase in the local temperature will be observed. There are also very slow relaxation processes (CO_2 under $10{\cdot}6\,\mu m$ radiation). For these processes, cooling (at least temporarily) of the fluid occurs. In the first case, self-defocusing (thermal blooming) of the light beam takes place; in the second, a focusing effect is observed.

In recent years, a large number of publications have appeared which deal with the problems of light propagation through a fluid in general and, for obvious reasons, with the special case of $10{\cdot}6\,\mu m$ laser-beam transmission in particular. We mention a few typical references: Livingston (1971), Hayes (1971, 1972), Wallace & Camac (1970), Wood, Camac & Gerry (1971), Hayes, Ulrich & Aitken (1972) and Gebhardt & Smith (1972). In these publications, optical considerations (the wave or ray approach) have been put in the foreground and, with the exception of Livingston's paper, the induced convective flow problem has essentially been ignored for various reasons.

In this paper, the objective will be different. For the case of steady-state natural convection induced by a continuous-wave light beam, emphasis will be put on the interaction between the propagation of light intensity and distribution of heat and the induced natural flow. We shall specifically be interested in the general functional dependence of the induced fluid flow and fluid temperature on the intensity distribution in space based on the determination of the intensity (Poynting flux) as a function of the spatial co-ordinates. In particular, we shall numerically determine the maximal flow velocity and the maximal air temperature as functions of the initial intensity for vertical $10{\cdot}6\,\mu m$ CO_2 laser light propagation in air.

It is assumed that the fluid remains incompressible (Mach number very small compared with unity) and that the flow remains laminar. Furthermore, it is assumed that initially a parallel beam of light of a given axially symmetric Gaussian intensity distribution is propagating vertically upwards (in the z direction) into a still homogeneous fluid of constant density ρ ($\mathrm{g\,cm^{-3}}$). The intensity will be restricted to reasonably small values, of the order of $10^4\,\mathrm{W\,cm^{-2}}$, say, to avoid ionization of the fluid (Steverding 1972) and nonlinear effects which would make some of the physical parameters strongly dependent on temperature. The fluid may be characterized by its viscosity μ ($\mathrm{g\,cm^{-1}\,s^{-1}}$), kinematic viscosity $\nu = \mu\rho^{-1}$ ($\mathrm{cm^2\,s^{-1}}$), specific heat at constant pressure c_p ($\mathrm{cal\,g^{-1}\,{}^\circ C^{-1}}$), conductivity k ($\mathrm{cal\,cm^{-1}\,s^{-1}\,{}^\circ C^{-1}}$) and absorption coefficient α ($\mathrm{cm^{-1}}$), which are all considered as constants. It is also assumed that the molecular relaxation process is rapid so that the absorbed energy is instantaneously transformed into heat.

The interaction process we are going to investigate may be termed 'weak' in contrast to 'strong' interactions which cause physical changes in the fluid, changes in the absorption coefficient and the occurrence of turbulent flow. Physical changes will be completely disregarded in our investigation and initial intensities will be restricted to values which make the fluid velocity remain in the laminar domain and keep the absorption coefficient constant. Two basic papers which deal with the onset of induced turbulence deserve mention here: Chodzko & Lin (1971) and Wagner & Marburger (1971). The former is also concerned with temperature-dependent absorption.

The co-ordinate system is cylindrical (z, r, ϕ), all functions being independent of ϕ because of the assumed axial symmetry of the intensity distribution. The intensity initiates from the plane $z = 0$ and propagates in the $+z$ direction. The initial intensity in the plane $z = 0$ is assumed to be of the form $X_0 \exp(-r^2 a_0^{-2})$ with X_0 being in W cm^{-1}. The length a_0 (cm) is the half-width of the Gaussian intensity profile at $z = 0$. If we are thinking of light being emitted from a small aperture, a_0 may be considered as small to obtain an intensity distribution that is sharply peaked in the neighbourhood of the aperture.

If the initial intensity distribution is indeed sharply peaked, it seems natural to try to apply boundary-layer theory to describe the coupled process of distribution of intensity, temperature and fluid flow. This approach is similar to that taken in the description of jets, for example, where there are also no solid boundaries (Pai 1954; Schlichting 1960).

2. Boundary-layer theory approach

Let $v = v(r,z)$ (cm s^{-1}) and $w = w(r,z)$ (cm s^{-1}) be the horizontal (r direction) and vertical (z direction) components of the fluid velocity vector. The steady-state Prandtl boundary-layer equations for the conservation of mass (continuity) and momentum (Navier–Stokes) are

$$\frac{1}{r}\frac{\partial(rv)}{\partial r} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$v\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + g\theta, \tag{2}$$

where g (cm s^{-2}) is the acceleration due to gravity and

$$\theta = \theta(r,z) = (T - T_\infty)\, T_\infty^{-1}$$

represents the dimensionless temperature relative to the temperature T_∞ (°C) at infinity. The buoyancy force per unit volume is $\rho g \theta$.

The conservation of energy equation is

$$c_p \rho \left(v\frac{\partial \theta}{\partial r} + w\frac{\partial \theta}{\partial z}\right) = k\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right) + \alpha T_\infty^{-1} I, \tag{3}$$

in which I (W cm^{-2}) represents the local light intensity for $z \geqslant 0$.

The pressure gradient dp/dz has been neglected as is customary in applications of boundary-layer theory because of the assumption that the constant pressure of the surrounding fluid impresses itself on the boundary-layer flow.

The affine transformation

$$\xi = r a^{-1}(z)$$

will be introduced; here the positive function $a(z)$ has the dimension of length (cm) to render ξ dimensionless. Equation (1) can then be satisfied by means of the stream function (dimensions cm^3 s^{-1})

$$\Phi(r,z) = \nu h(z) f(\xi) \tag{4}$$

(with h in cm), which gives the velocities

$$w = r^{-1}\,\partial\Phi/\partial r = \nu h a^{-2}\xi^{-1}f', \tag{5}$$

$$v = -r^{-1}\,\partial\Phi/\partial z = -\nu h' a^{-1}\xi^{-1}f + \nu h a^{-2}a'f', \tag{6}$$

where a prime signifies differentiation with respect to z for the functions a and h and differentiation with respect to ξ for the function f. The use of only one differentiation symbol will not cause confusion since the functions a, h, f and later F depend on one variable only. It will greatly simplify the outward appearance of the following differential equations.

Using now (5) and (6) in (2), we obtain

$$a^{-4}h[\xi^{-1}f'''' - \xi^{-2}f'' + h'\xi^{-2}ff'' + (2a^{-1}a'h - h')\xi^{-2}f'^2 - h'\xi^{-3}ff' + \xi^{-3}f' + g\nu^{-2}a^4h^{-1}\theta] = 0. \tag{7}$$

This gives us a differential equation for the function $f(\xi)$. For this equation to specify $f(\xi)$, its coefficients must be independent of z. This requires, first of all, that the dimensionless temperature θ be of the form

$$\theta(r,z) = H(z)\,F(\xi), \tag{8}$$

where H and F are dimensionless. Since there are coefficients that are already independent of z, it follows next that all those coefficients that appear as functions of z must be identically constant. Therefore, we have the conditions

$$\text{(i)} \qquad H = \alpha_1 a^{-4}h, \quad \alpha_1 = \text{constant} \quad (\text{cm}^3), \tag{9}$$

and, for the third and fourth coefficients,

$$\text{(ii)} \quad h' = \alpha_2 \geqslant 0, \quad \alpha_2 = \text{constant}, \tag{10}$$

$$\text{(iii)} \quad a^{-1}a'h = \alpha_3, \quad \alpha_3 = \text{constant}, \tag{11}$$

with α_2 and α_3 dimensionless.

Equation (3) leads to

$$a^{-2}H[F'' + (1 + Ph'f)\xi^{-1}F' - PhH^{-1}H'\xi^{-1}f'F + \alpha(kT_\infty)^{-1}a^2H^{-1}I] = 0, \tag{12}$$

where $P = \nu\kappa^{-1}$ = Prandtl number and $\kappa = k(c_p\rho)^{-1}$ = thermal diffusivity (in $\text{cm}^2\,\text{s}^{-1}$). For (12) to specify $F(\xi)$, the coefficients must be independent of z. This requires that the intensity I be expressible as a product of a function of z and a function of ξ. Therefore we set

$$I(r,z) = e^{-\alpha z}X(z)\,Y(\xi),$$

where the factor $\exp(-\alpha z)$ is due to absorption by the fluid, $X(z)$ (W cm^2) is due to changes in the beam diameter and $Y(\xi)$ is the (dimensionless) intensity form factor. Since $Y(\xi)$ must agree with the initial form factor $\exp(-r^2a_0^2)$ for $z = 0$, we have

$$Y(\xi) = \exp(-\xi^2), \quad \xi = ra^{-1}(z).$$

Now, to make all coefficients of (12) independent of z, it is necessary that

$$\text{(iv)} \qquad e^{-\alpha z}X = \alpha_4 Ha^{-2}, \quad \alpha_4 = \text{constant} \quad (\text{W}), \tag{13}$$

$$\text{(v)} \quad hH^{-1}H' = \alpha_5, \quad \alpha_5 = \text{constant}, \tag{14}$$

with α_5 dimensionless. For easy reference, we collect here the constants $\alpha_1, \ldots, \alpha_5$ which appear now as coefficients in the differential equations (7) and (12) for $f(\xi)$ and $F(\xi)$:

$$\alpha_1 = a^4h^{-1}H \ (\text{cm}^3), \quad \alpha_2 = h' \ (\text{dimensionless}), \quad \alpha_3 = a^{-1}a'h \ (\text{dimensionless}),$$
$$\alpha_4 = e^{-\alpha z}a^2H^{-1}X \ (\text{W}), \quad \alpha_5 = hH^{-1}H' \ (\text{dimensionless}).$$

They will be specified next.

3. Evaluation of the constants

Conditions (11) and (14), considered as differential equations for $a(z)$ and $H(z)$, respectively, lead immediately to

$$a(z) = a_0 \exp\left(\alpha_3 \int_0^z h^{-1}\,d\zeta\right), \quad a_0 > 0, \tag{15}$$

$$H(z) = H_0 \exp\left(\alpha_5 \int_0^z h^{-1}\,d\zeta\right), \quad H_0 > 0. \tag{16}$$

Relation (9), as $z \to 0$, shows that

$$H_0 = \alpha_1 h_0 a_0^{-4}, \quad h_0 = h(0) > 0.$$

If the constant $h' = \alpha_2$ [see (10)] were positive, then h would be a linear function of z, which, according to (15) and (16), would imply that a and H would be rational functions of z. From our original assumption that light energy is instantaneously transformed into heat it follows that no self-focusing of the light beam can occur. In other words, the function $X(z)$ appearing in the expression for the intensity I must be a non-increasing function of z. According to (13), this requires that Ha^{-2} decreases at least as fast as $\exp \alpha z$ increases. Therefore, H and a cannot both be rational functions. Consequently, the constant $h' = \alpha_2$ must be zero, which makes $h(z)$ identically constant, $h(z) \equiv h_0 > 0$, and hence, according to (15) and (16),

$$a(z) = a_0 \exp(\alpha_3 h_0^{-1} z), \quad H(z) = \alpha_1 h_0 a_0^{-4} \exp(\alpha_5 h_0^{-1} z). \tag{17}$$

Furthermore, it follows now from (9) that

$$H(z) = \alpha_1 h_0 a_0^{-4} \exp(-4\alpha_3 h_0^{-1} z). \tag{18}$$

Therefore, if we compare this expression for H with that in (17), we see that $\alpha_5 = -4\alpha_3$. Next it follows from (13) and (18), as $z \to 0$, that

$$\alpha_4 = a_0^2 X_0 H_0^{-1} = (\alpha_1 h_0)^{-1} a_0^6 X_0, \quad X_0 = X(0).$$

Finally, let us eliminate the constant α_3. To this end, we observe the following. If no thermally induced modification of the light beam occurred, the intensity would be reduced by absorption only, i.e. at $z > 0$, it would be

$$I^* = e^{-\alpha z} X_0 Y(r a_0^{-1})$$

and hence the energy passing per unit time through a plane parallel to $z = 0$ would be

$$J^* = \int_0^{2\pi} \int_0^{\infty} I^* r\,dr\,d\phi.$$

If we assume that thermal modification of the beam does not greatly affect the total flow of energy but only its local distribution, it follows that

$$J^* = J = \int_0^{2\pi}\int_0^{\infty} Ir\,dr\,d\phi, \quad I = e^{-\alpha z}X(z)\,Y(\xi).$$

The equality $J = J^*$ implies then that

$$a^2(z)\,X(z) \equiv a_0^2 X_0 = \text{constant}.$$

As (13) shows, this relation means that $H(z)\exp\alpha z$ is identically constant. Therefore, using (18), we see that

$$\alpha_3 = \tfrac{1}{4}\alpha h_0. \tag{19}$$

We are thus left with the constants $\alpha_1, \alpha_2 = 0$, $\alpha_3 = \frac{1}{4}\alpha h_0$, $\alpha_4 = (\alpha_1 h_0)^{-1} a_0^6 X_0$ and $\alpha_5 = -\alpha h_0$. It will turn out that α_1 and h_0 are irrelevant, so that only the physically meaningful parameters α, the absorption coefficient, a_0, the initial intensity form factor parameter, and X_0, the initial intensity at $r = 0$, remain.

4. General results

We are now in a position to discuss the main results. According to (17) and (19), the boundary-layer thickness is characterized by

$$a(z) = a_0 \exp(\tfrac{1}{4}\alpha z), \tag{20}$$

i.e. it increases exponentially with increasing distance from the plane of light emission.

The temperature decreases exponentially in the direction of light propagation as indicated by the function

$$H(z) = \alpha_1 h_0 a_0^{-4} \exp(-\alpha z),$$

which follows from (18) together with (19).

The intensity of light decreases in the propagation direction according to

$$e^{-\alpha z}X(z) = X_0 \exp(-\tfrac{3}{2}\alpha z)$$

and this decrease is partly due to the defocusing effect, which is characterized by

$$X(z) = X_0 \exp(-\tfrac{1}{2}\alpha z).$$

To complete the investigation it is now necessary to return to the differential equations (7) and (12) for $f(\xi)$ and $F(\xi)$. Using the values of the parameters $\alpha_1, \ldots, \alpha_5$ given near the end of §3, we obtain from (7) and (12)

$$\xi^{-1}f''' - \xi^{-2}f'' + \tfrac{1}{2}\alpha h_0 \xi^{-2} f'^2 + \xi^{-3} f' + \alpha_1 g\nu^{-2} F = 0,$$

$$F'' + \xi^{-1}F' + \alpha h_0 P \xi^{-1} f' F + \alpha (kT_\infty)^{-1} (\alpha_1 h_0)^{-1} \alpha_0^6 X_0\, Y = 0.$$

To eliminate the constants α_1 and h_0, we introduce new functions $q(\xi)$ and $Q(\xi)$ by setting

$$f(\xi) = (\alpha h_0)^{-1} q(\xi), \quad F(\xi) = (\alpha g \alpha_1 h_0)^{-1} \nu^2 Q(\xi) \tag{21}$$

and arrive at the following coupled system of nonlinear ordinary differential equations for $q(\xi)$ and $Q(\xi)$:

$$\xi^{-1}q''' - \xi^{-2}q'' + \tfrac{1}{2}\xi^{-2}q'^2 + \xi^{-3}q' + Q = 0, \tag{22}$$

$$Q'' + \xi^{-1}Q' + P\xi^{-1}q'Q + \sigma Y = 0, \tag{23}$$

with the driving term $Y(\xi) = \exp(-\xi^2)$ and the dimensionless parameter

$$\sigma = (k\nu^2 T_\infty)^{-1}\,\alpha^2 g a_0^6 X_0. \tag{24}$$

As a consequence of the transformations (21), the stream function Φ, given by (4), and the dimensionless temperature θ, given by (8), take the forms

$$\Phi(r,z) = \nu\alpha^{-1}q(\xi) \tag{25}$$

and

$$\theta(r,z) = \nu^2(\alpha g a_0^4)^{-1}\exp(-\alpha z)\,Q(\xi),$$

where $\xi = ra^{-1}(z)$. Consequently, if we consider (5) and (6), we obtain from (25) the flow velocity components

$$v(r,z) = \nu\alpha^{-1}a^{-2}(z)\,a'(z)\,q'(\xi),$$

$$w(r,z) = \nu\alpha^{-1}a^{-2}(z)\,\xi^{-1}q'(\xi),$$

or, if we use (20),

$$v(r,z) = \tfrac{1}{4}\nu a_0^{-1}\exp(-\tfrac{1}{4}\alpha z)\,q'(\xi), \tag{26}$$

$$w(r,z) = \nu\alpha^{-1}a_0^{-1}\exp(-\tfrac{1}{2}\alpha z)\,\xi^{-1}q'(\xi). \tag{27}$$

Since no closed-form solutions of the system (22) and (23) are known, we are going to introduce an approximate solution for sufficiently small $|\xi|$ since we are essentially interested only in the flow velocity and the fluid temperature near $r = 0$ under the boundary conditions

$$q(\xi),\ q'(\xi),\ Q'(\xi) \to 0 \quad \text{as} \quad \xi \to 0. \tag{28}$$

The limiting relation concerning q' follows directly from (26) since, for symmetry reasons, $v(r,z) \to 0$ as $r \to 0$. We also see from (27) that q' must behave like a linear function near $\xi = 0$ because $w(r,z)$ must approach a finite and positive limit as $r \to 0$. The condition on q follows from the definition of the stream function Φ given in (4) if we also consider (21). It would also follow from the general expression for the horizontal velocity component v as given in (6). The condition on Q is a consequence of the assumption that the initial intensity distribution is an even function of ξ, so that $\partial\theta/\partial r \to 0$ as $r \to 0$. We then consider (8) and (21).

It is now useful to look at the functions

$$q^*(\xi) = \beta(1 - e^{-\xi^2}), \quad 0 \leqslant \beta < +\infty, \tag{29}$$

and

$$Q^*(\xi) = \gamma\, e^{-\xi^2}, \quad 0 \leqslant \gamma < +\infty. \tag{30}$$

They satisfy the boundary conditions (28) and they can be made to be ϵ-approximate solutions of (22) and (23) in some neighbourhood of $\xi = 0$ if the coefficients β and γ are suitably chosen. To see this, we evaluate the left-hand sides of (22) and (23) in terms of q^* and Q^*. This leads to the functions

$$\phi_1(\xi;\beta,\gamma) = (-8\beta + 2\beta^2 e^{-\xi^2} + 8\beta\xi^2 + \gamma)\,e^{-\xi^2},$$

$$\phi_2(\xi;\beta,\gamma) = (-4\gamma + 2P\beta\gamma\, e^{-\xi^2} + 4\gamma\xi^2 + \sigma)\,e^{-\xi^2}.$$

Here we let $\xi \to 0$ and obtain

$$\phi_1(0; \beta, \gamma) = -8\beta + 2\beta^2 + \gamma,$$
$$\phi_2(0; \beta, \gamma) = -4\gamma + 2P\beta\gamma + \sigma.$$

Then, if β^* and γ^* are the solutions of the system of algebraic equations

$$-8\beta + 2\beta^2 + \gamma = 0, \quad -4\gamma + 2P\beta\gamma + \sigma = 0, \tag{31 a, b}$$

the functions $\phi_1(\xi; \beta^*, \gamma^*)$ and $\phi_2(\xi; \beta^*, \gamma^*)$ can be made absolutely less than any given $\epsilon > 0$ if $|\xi|$ is sufficiently small.

Equations (31) permit us to determine the parameters β and γ occurring in the approximations q^* and Q^* for q and Q, respectively, as functions of the non-negative parameter σ, which, according to (24), combines the physically relevant parameters, in particular the peak intensity X_0 of the light beam. Equation (31 *a*) leads to

$$\beta = 2 - (4 - \tfrac{1}{2}\gamma)^{\frac{1}{2}}. \tag{32}$$

The negative sign in front of the square root must be chosen because if $\gamma = 0$, i.e. if there is no increase in temperature, there will be no fluid flow. Equation (31 *b*) leads then to the third-order equation

$$p(\gamma; \sigma) = 2P^2\gamma^3 - 16(2P-1)\gamma^2 - 8(1-P)\sigma\gamma + \sigma^2 = 0 \tag{33}$$

for γ as a function of σ. For $\sigma = 0$, $p(\gamma; 0)$ has the zeros

$$\gamma_{1,2} = 0 \quad \text{and} \quad \gamma_3 = 8(2P-1)P^{-2}.$$

The zero γ_3 is positive provided that the Prandtl number P is greater than $\frac{1}{2}$. Thus, for $\sigma = 0$, i.e. for $X_0 = 0$, $\gamma^* = 0$ and, according to (32), $\beta^* = 0$ as they should be in the absence of a driving term in (23).

If $\sigma \in (0, +\infty)$, an application of Sturm's theorem (cf., for example, Lehnigk 1966, p. 112) shows that $p(\gamma; \sigma > 0)$ has one negative zero $\gamma_1 < 0$, and two positive zeros γ_2 and γ_3, $0 < \gamma_2 < \gamma_3$. The smaller one of these, γ_2, i.e. the one that branches to the right from the double zero at the origin for $\sigma = 0$, is, for continuity reasons, the one we have to use. Thus, $\gamma^* = \gamma^*(\sigma) = \gamma_2$ for $\sigma \in [0, +\infty)$ and, according to (32),

$$\beta^* = \beta^*(\sigma) = 2 - (4 - \tfrac{1}{2}\gamma^*(\sigma))^{\frac{1}{2}} \quad \text{for} \quad \sigma > 0, \quad \gamma^*(\sigma) \leqslant 8 \tag{34}$$

are the parameters that make the functions q^* and Q^* of (29) and (30) ϵ-approximate solutions of (22) and (23) in some neighbourhood of $\xi = 0$. Both β^* and γ^* increase monotonically as σ increases. Of course, in practice σ is not allowed to increase indefinitely because of the original assumption of an incompressible fluid, which requires the Mach number to be small compared with unity. In our problem, this requires that the maximal vertical flow velocity $w(r, z)$, which occurs at $r = z = 0$ [see (27)], be small compared with the velocity c of sound in the fluid. As a matter of fact, $\sigma < 10^{-1}$ in the example to be considered in the next section.

The smallness of σ makes it possible to obtain good approximate expressions for γ^* and β^* as functions of σ. For, if σ is small, we may set $\gamma = \lambda\sigma$ in (33) and neglect the cubic term. This gives a quadratic equation for λ, the positive root of which we need. It turns out that this positive root is $\frac{1}{4}$, independent of the Prandtl number. Therefore,

$$\gamma^* = \gamma^*(\sigma) = \tfrac{1}{4}\sigma, \quad \text{for small } \sigma > 0,$$

and, according to (34),

$$\beta^* = \beta^*(\sigma) = 2-(4-\tfrac{1}{8}\sigma)^{\frac{1}{2}} \quad \text{for small } \sigma > 0. \tag{35}$$

As a function of σ, the maximal vertical flow velocity, which occurs at $r = z = 0$, now becomes

$$w(0,0) = 2\nu\alpha^{-1}a_0^{-2}[2-(4-\tfrac{1}{8}\sigma)^{\frac{1}{2}}] \quad \text{for small } \sigma > 0. \tag{36}$$

We have used here (27) with $q(\xi)$ replaced by $q^*(\xi)$ as given by (29) and with $\beta = \beta^*$ as given by (35).

We can now also indicate the dependence of the stream function and the dimensionless temperature on σ for small σ:

$$\begin{aligned} \Phi(r,z) &= \nu\alpha^{-1}[2-(4-\tfrac{1}{8}\sigma)^{\frac{1}{2}})]\,(1-e^{-\xi^2}), \\ \theta(r,z) &= \tfrac{1}{4}\nu^2(\alpha g a_0^4)^{-1}\,\sigma\, e^{-\alpha z-\xi^2}, \end{aligned} \tag{37}$$

where $\xi = ra^{-1}(z)$.

5. Numerical results

In this section, we wish to specialize our general results for the case of convective flow induced by CO_2 laser heating at 10·6 μm in air. We use the following numerical values.

Kinematic viscosity	$\nu = 15\times 10^{-2}\,\text{cm}^2\,\text{s}^{-1}$.
Conductivity	$k = 6\times 10^{-5}\,\text{cal s}^{-1}\,\text{cm}^{-1}\,{}^\circ\text{C}^{-1}$.
Velocity of sound	$c = 33\times 10^{3}\,\text{cm s}^{-1}$.
Air temperature at infinity	$T_\infty = 20\,{}^\circ\text{C}$.
Absorption coefficient	$\alpha = 2\times 10^{-6}\,\text{cm}^{-1}$.
Acceleration due to gravity	$g = 10^{3}\,\text{cm s}^{-2}$.

Of primary interest in this investigation are the induced convective fluid flow and the fluid temperature as functions of the half-width of the intensity profile at $z = 0$ and the intensity X_0 at $r = z = 0$.

The vertical velocity component $w(0,0)$ at $r = z = 0$ of the fluid flow as given by (36) is of particular interest. To avoid calculating values for $w(0,0)$ too large for the assumption of laminar flow, we choose a critical Reynolds number $R_{\text{crit}} = 2\times 10^3$. We define the Reynolds number R of the flow at $z = 0$ as $R = w(0,0)\,\delta\nu^{-1}$, where $\delta = 2a_0$ is the boundary-layer thickness at $z = 0$. Then, from (36) and (35), $R = 4\alpha^{-1}a_0^{-1}\beta^*$. The condition $R \leqslant R_{\text{crit}} = 2\times 10^3$ leads to $\beta^* \leqslant \frac{1}{2}10^3\alpha a_0 = 10^{-3}a_0 = \beta^*_{\max}$ and, consequently, $w(0,0) \leqslant 10^3\nu a_0^{-1} = 150a_0^{-1}$ (cm s^{-1}). The laminar-flow boundary is the dashed hyperbola in figure 1. If we take the Mach number to be not greater than 10^{-1}, the laminar-flow bound is much less than the incompressibility bound.

a_0 (cm)	$X_{0\max}$ (W cm^{-2})	a_0 (cm)	$X_{0\max}$ (W cm^{-2})
0·6	11 574	1·6	86
0·7	5 354	1·7	64
0·8	2 746	1·8	48
0·9	1 524	1·9	36
1·0	900	2·0	28
1·1	558	2·2	17.4
1·2	362	2·4	11.4
1·3	242	2·6	7.6
1·4	168	2·8	5.2
1·5	118	3·0	3.8

TABLE 1. Maximal intensities for laminar flow as function of half-width a_0 of intensity profile at $z = 0$

To determine next maximal values for X_0 as a function of a_0 for laminar flow conditions, we investigate the inequality

$$\beta^* = 2-(4-\tfrac{1}{8}\sigma)^{\frac{1}{2}} \leqslant \beta^*_{\max} = 10^{-3}a_0,$$

with σ as function of a_0 and X_0 given by (24). This inequality leads to

$$X_0 \leqslant 225a_0^{-5}(4-10^{-3}a_0) \quad (\mathrm{W\,cm^{-2}}).$$

If a_0 is taken to be sufficiently small, this inequality may be simplified to

$$X_0 \leqslant 900a_0^{-5} \quad (\mathrm{W\,cm^{-2}}).$$

In table 1, the maximal values $X_{0\max} = 900a_0^{-5}$ are displayed for $0{\cdot}6 \leqslant |a_0| \leqslant 3{\cdot}0$.

It is also useful to display the dependence of the parameter σ, given by (24), on a_0 and X_0. With the numerical values chosen at the beginning of this section, we have

$$\sigma = (6750)^{-1}a_0^6X_0.$$

In table 2, σ is given as function of a_0 for $0{\cdot}4 \leqslant |a_0| \leqslant 2{\cdot}8$ and for five typical values of X_0 up to the order of $10^4\,\mathrm{W\,cm^{-2}}$.

Figure 1 shows the dependence of the vertical velocity $w(0,0)$ given by (36) on a_0 for the five X_0 values used in table 2. The laminar-flow region is below the dashed hyperbola. Above it is the turbulent region.

Using (37) and the relation

$$\Delta T = T_\infty\theta(r,z) \quad (^\circ\mathrm{C})$$

we can now also calculate the temperature difference ΔT at $r = z = 0$. With the numerical parameters chosen we have

$$\theta(0,0) = 281\times10^{-2}\sigma a_0^{-4}.$$

Figure 2 exhibits ΔT as function of a_0 for the intensities X_{0i} ($i = 1,2,3,4$). For X_{05} and $|a_0| = 0{\cdot}4$ and $|a_0| = 0{\cdot}6$, we have $\Delta T = 3{\cdot}70\,^\circ$C and $\Delta T = 8{\cdot}32\,^\circ$C, respectively. We observe that a relatively small temperature increase is caused

	X_{01}	X_{02}	X_{03}	X_{04}	X_{05}
Watts cm^{-2} ...	5·2	17·4	86	900	11 574
cal sec^{-1} cm^{-2}...	1·25	4·18	20·64	216	2 778
a_0 (cm)	$10^5\,\sigma_1$	$10^5\,\sigma_2$	$10^5\,\sigma_3$	$10^5\,\sigma_4$	$10^5\,\sigma_5$
0·4	0·08	0·25	1·3	13·2	168·5
0·6	0·86	2·9	14·4	150	1919·6
0·8	4·8	16	80	840	—
1·0	18·4	61·8	300	3200	—
1·2	56	183	900	—	—
1·4	138	465	2300	—	—
1·6	310	1038	5140	—	—
1·8	626	2102	—	—	—
2·0	1178	3956	—	—	—
2·2	2086	7000	—	—	—
2.4	3516	—	—	—	—
2·6	5684	—	—	—	—
2·8	8866	—	—	—	—

TABLE 2. The parameter σ as function of a_0 and X_0

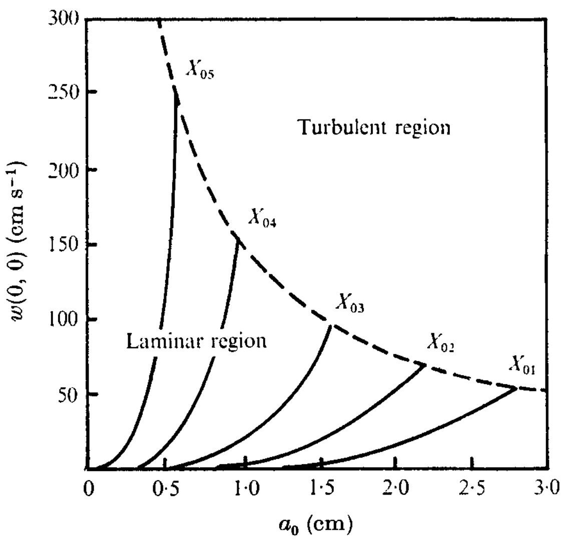

FIGURE 1. Vertical flow velocity $w(0, 0)$ at the origin as function of intensity-profile half-width a_0 at $z = 0$ for intensities $X_{01} = 5·2$, $X_{02} = 17·4$, $X_{03} = 86$, $X_{04} = 900$, $X_{05} = 11\,574$ W cm^{-2}.

by the light propagation. The values of ΔT for $|a_0| = 1$ can, of course, be obtained directly from the differential equation (3) for $r = z = 0$ with the numerical value 24×10^{-2} cal g^{-1} °C^{-1} for the specific heat c_p of air.

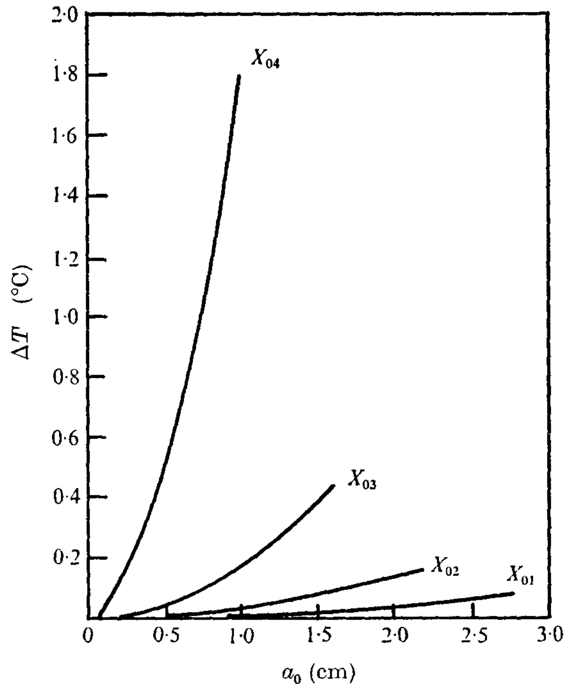

FIGURE 2. Temperature difference ΔT at the origin as function of intensity-profile half-width a_0 at $z = 0$ for intensities $X_{01} = 5{\cdot}2$, $X_{02} = 17{\cdot}4$, $X_{03} = 86$, $X_{04} = 900$ W cm^{-2}.

6. Concluding remarks

Under simplifying assumptions, we have calculated the free convective flow field and temperature distribution induced by vertically propagating light in a mildly absorbing fluid. We have explicitly determined the stream function, which leads to the flow velocity components, and the dimensionless fluid temperature under boundary-layer assumptions in terms of the physically relevant parameters and as functions of the spatial co-ordinates.

Assumptions closer to reality might include a pre-existing flow transverse to the direction of light propagation or horizontal propagation instead of vertical propagation.

With respect to the forced convection problem it should be observed that the buoyancy forces become negligibly small if the transverse flow velocity is sufficiently large and that, consequently, approaches different from the one used in this paper become appropriate. We refer again to the papers of Livingston and Wallace & Camac, for example. It should also be pointed out that the constant (forcing) term appearing in the appropriate fluid mechanical equation due to a constant transverse flow ruins the similarity transform approach.

An investigation of steady-state horizontal light propagation by means of boundary-layer theory and similarity transform methods is in progress and we shall report on its results as soon as it has been completed. This problem is, of course, much more complicated than the idealized one we have studied in the present paper; first, because of the different geometric situation, which requires that the stream and temperature functions be expressed as functions of three co-ordinates, and, second, because the induced free convective flow is here perpendicular to the direction of light propagation, which causes typical deformations in the contours of constant intensity (see Livingston 1971).

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